HL Paper 2

Find the Cartesian equation of plane *Π* containing the points $\mathrm{A}\,(6,\ 2,\ 1)$ and $\mathrm{B}\,(3,\ -1,\ 1)$ and perpendicular to the plane $x + 2y - z - 6 = 0$.

Given that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \neq \boldsymbol{0}$ prove that $\boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ where *s* is a scalar.

Consider the points P(−3, −1, 2) and Q(5, 5, 6).

a. Find a vector equation for the line, L_1, which passes through the points P and Q. [3]

The line L_2 has equation

$$r = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}.$$

b. Show that L_1 and L_2 intersect at the point R(1, 2, 4). [4]

c. Find the acute angle between L_1 and L_2. [3]

d. Let S be a point on L_2 such that $\left|\overrightarrow{\mathrm{RP}}\right| = \left|\overrightarrow{\mathrm{RS}}\right|$. [6]

Show that one of the possible positions for S is $\mathrm{S}_1(-4, 0, 4)$ and find the coordinates of the other possible position, S_2.

e. Let S be a point on L_2 such that $\left|\overrightarrow{\mathrm{RP}}\right| = \left|\overrightarrow{\mathrm{RS}}\right|$. [4]

Find a vector equation of the line which passes through R and bisects $\mathrm{P\hat{R}S_1}$.

The points P(−1, 2, − 3), Q(−2, 1, 0), R(0, 5, 1) and S form a parallelogram, where S is diagonally opposite Q.

a. Find the coordinates of S. [2]

b. The vector product $\overrightarrow{\mathrm{PQ}} \times \overrightarrow{\mathrm{PS}} = \begin{pmatrix} -13 \\ 7 \\ m \end{pmatrix}$. Find the value of m . [2]

c. Hence calculate the area of parallelogram PQRS. [2]

d. Find the Cartesian equation of the plane, $\prod_1$, containing the parallelogram PQRS. [3]

e. Write down the vector equation of the line through the origin (0, 0, 0) that is perpendicular to the plane $\prod_1$. [1]

f. Hence find the point on the plane that is closest to the origin. [3]

g. A second plane, $\prod_2$, has equation $x - 2y + z = 3$. Calculate the angle between the two planes. [4]

The vector equation of line l is given as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

Find the Cartesian equation of the plane containing the line l and the point A$(4, -2, 5)$.

Consider the vectors $\boldsymbol{a} = \sin(2\alpha)\boldsymbol{i} - \cos(2\alpha)\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \cos\alpha\boldsymbol{i} - \sin\alpha\boldsymbol{j} - \boldsymbol{k}$, where $0 < \alpha < 2\pi$.

Let θ be the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) Express $\cos\theta$ in terms of α.

(b) Find the acute angle α for which the two vectors are perpendicular.

(c) For $\alpha = \frac{7\pi}{6}$, determine the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ and comment on the geometrical significance of this result.

The lines l_1 and l_2 are defined as

$l_1 : \frac{x-1}{3} = \frac{y-5}{2} = \frac{z-12}{-2}$

$l_2 : \frac{x-1}{8} = \frac{y-5}{11} = \frac{z-12}{6}$.

The plane π contains both l_1 and l_2.

a. Find the Cartesian equation of π. [4]

b. The line l_3 passing through the point $(4,\ 0,\ 8)$ is perpendicular to π. [4]

Find the coordinates of the point where l_3 meets π.

(a) If $a = 4$ find the coordinates of the point of intersection of the three planes.

(b) (i) Find the value of a for which the planes do not meet at a unique point.

(ii) For this value of a show that the three planes do not have any common point.

The points A and B have position vectors $\overrightarrow{\mathrm{OA}} = \begin{Bmatrix} 1 \\ 2 \\ -2 \end{Bmatrix}$ and $\overrightarrow{\mathrm{OB}} = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$.

a. Find $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}}$. [2]

b. Hence find the area of the triangle OAB. [2]

The points A and B have coordinates (1, 2, 3) and (3, 1, 2) relative to an origin O.

a. (i) Find $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}}$. [5]

(ii) Determine the area of the triangle OAB.

(iii) Find the Cartesian equation of the plane OAB.

b. (i) Find the vector equation of the line L_1 containing the points A and B. [7]

(ii) The line L_2 has vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

Determine whether or not L_1 and L_2 are skew.

Consider the planes $\pi_1 : x - 2y - 3z = 2$ and $\pi_2 : 2x - y - z = k$.

a. Find the angle between the planes $\pi_1$and π_2 . [4]

b. The planes π_1 and π_2 intersect in the line L_1 . Show that the vector equation of [5]

L_1 is $r = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

c. The line L_2 has Cartesian equation $5 - x = y + 3 = 2 - 2z$. The lines L_1 and L_2 intersect at a point X. Find the coordinates of X. [5]

d. Determine a Cartesian equation of the plane π_3 containing both lines L_1 and L_2 . [5]

e. Let Y be a point on L_1 and Z be a point on L_2 such that XY is perpendicular to YZ and the area of the triangle XYZ is 3. Find the perimeter of the triangle XYZ. [5]

Find the vector equation of the line of intersection of the three planes represented by the following system of equations.

$$2x - 7y + 5z = 1$$

$$6x + 3y - z = -1$$

$$-14x - 23y + 13z = 5$$

Two submarines A and B have their routes planned so that their positions at time t hours, $0 \le t < 20$, would be defined by the position vectors $\boldsymbol{r}_A = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -0.15 \end{pmatrix}$ and $\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix}$ relative to a fixed point on the surface of the ocean (all lengths are in kilometres).

To avoid the collision submarine B adjusts its velocity so that its position vector is now given by

$$\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}.$$

a. Show that the two submarines would collide at a point P and write down the coordinates of P. [4]

b.i. Show that submarine B travels in the same direction as originally planned. [1]

b.ii. Find the value of t when submarine B passes through P. [2]

c.i. Find an expression for the distance between the two submarines in terms of t. [5]

c.ii. Find the value of t when the two submarines are closest together. [2]

c.iii. Find the distance between the two submarines at this time. [1]

The points A, B and C have the following position vectors with respect to an origin O.

$$\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

$$\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

$$\overrightarrow{OC} = \boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$$

The plane Π_2 contains the points O, A and B and the plane Π_3 contains the points O, A and C.

a. Find the vector equation of the line (BC). [3]

b. Determine whether or not the lines (OA) and (BC) intersect. [6]

c. Find the Cartesian equation of the plane Π_1, which passes through C and is perpendicular to $\overrightarrow{OA}$. [3]

d. Show that the line (BC) lies in the plane Π_1. [2]

e. Verify that $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2. [3]

f. Find a vector perpendicular to the plane Π_3. [1]

g. Find the acute angle between the planes Π_2 and Π_3. [4]

OACB is a parallelogram with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$, where ***a*** and ***b*** are non-zero vectors.

a. Show that [4]

(i) $\left|\overrightarrow{OC}\right|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$;

(ii) $\left|\overrightarrow{AB}\right|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$.

b. Given that $\left|\overrightarrow{OC}\right| = \left|\overrightarrow{AB}\right|$, prove that OACB is a rectangle. [4]

Ed walks in a straight line from point $P(-1,\ 4)$ to point $Q(4,\ 16)$ with constant speed.

Ed starts from point P at time $t = 0$ and arrives at point Q at time $t = 3$, where t is measured in hours.

Given that, at time t, Ed's position vector, relative to the origin, can be given in the form, $r = a + tb$,

a. find the vectors a and b. [3]

b. Roderick is at a point $C(11,\ 9)$. During Ed's walk from P to Q Roderick wishes to signal to Ed. He decides to signal when Ed is at the closest point to C. [5]

Find the time when Roderick signals to Ed.

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a. Find the values of k for which the following system of equations has no solutions and the value of k for the system to have an infinite number of solutions. [5]

$$x - 3y + z = 3$$
$$x + 5y - 2z = 1$$
$$16y - 6z = k$$

b. Given that the system of equations can be solved, find the solutions in the form of a vector equation of a line, $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where the components of $\boldsymbol{b}$ are integers. [7]

c. The plane ÷ is parallel to both the line in part (b) and the line $\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-2}{0}$. [5]
Given that ÷ contains the point $(1, 2, 0)$, show that the Cartesian equation of ÷ is $16x + 24y - 11z = 64$.

d. The z-axis meets the plane ÷ at the point P. Find the coordinates of P. [2]

e. Find the angle between the line $\frac{x-2}{3} = \frac{y+5}{4} = \frac{z}{2}$ and the plane ÷ . [5]

(a) Find the coordinates of the point A on l_1 and the point B on l_2 such that $\overrightarrow{AB}$ is perpendicular to both l_1 and l_2.

(b) Find $|AB|$.

(c) Find the Cartesian equation of the plane $\prod$ which contains l_1 and does not intersect l_2 .

The angle between the vector $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and the vector $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + m\boldsymbol{k}$ is 30° .

Find the values of m.

Find the angle between the lines $\frac{x-1}{2} = 1 - y = 2z$ and $x = y = 3z$.

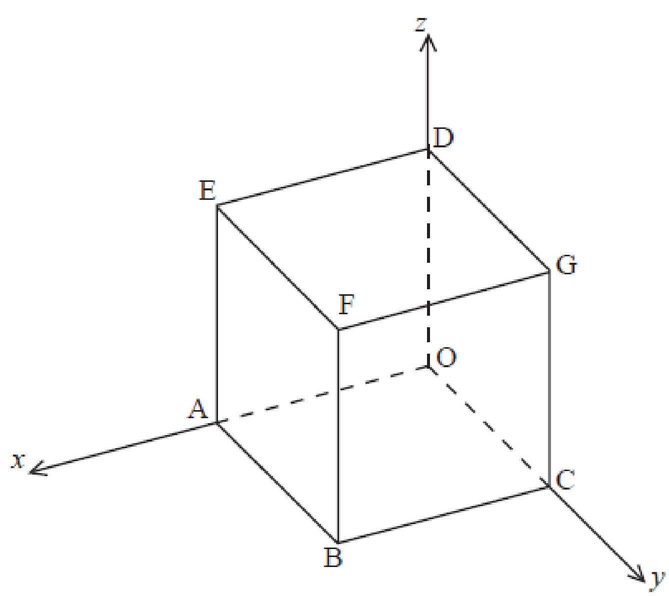
A curve is defined $x^2 - 5xy + y^2 = 7$.

a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y-2x}{2y-5x}$. [3]

b. Find the equation of the normal to the curve at the point $(6,\ 1)$. [4]

c. Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. [8]

The diagram shows a cube OABCDEFG.



Let O be the origin, (OA) the x-axis, (OC) the y-axis and (OD) the z-axis.

Let M, N and P be the midpoints of [FG], [DG] and [CG], respectively.

The coordinates of F are (2, 2, 2).

(a) Find the position vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$ and $\overrightarrow{OP}$ in component form.

(b) Find $\overrightarrow{MP} \times \overrightarrow{MN}$.

(c) **Hence,**

(i) calculate the area of the triangle MNP;

(ii) show that the line (AG) is perpendicular to the plane MNP;

(iii) find the equation of the plane MNP.

(d) Determine the coordinates of the point where the line (AG) meets the plane MNP.

OABCDE is a regular hexagon and ***a*** , ***b*** denote respectively the position vectors of A, B with respect to O.

a. Show that OC = 2AB . [2]

b. Find the position vectors of C, D and E in terms of ***a*** and ***b*** . [7]

Two planes Π_1 and Π_2 have equations $2x + y + z = 1$ and $3x + y - z = 2$ respectively.

a. Find the vector equation of L, the line of intersection of Π_1 and Π_2. [6]

b. Show that the plane Π_3 which is perpendicular to Π_1 and contains L, has equation $x - 2z = 1$. [4]

c. The point P has coordinates (−2, 4, 1) , the point Q lies on Π_3 and PQ is perpendicular to Π_2. Find the coordinates of Q. [6]

Consider the two planes

$\pi_1 : 4x + 2y - z = 8$

$\pi_2 : x + 3y + 3z = 3.$

Find the angle between π_1 and π_2, giving your answer correct to the nearest degree.

A ray of light coming from the point (−1, 3, 2) is travelling in the direction of vector $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and meets the plane $\pi : x + 3y + 2z - 24 = 0$.

Find the angle that the ray of light makes with the plane.

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are such that $\boldsymbol{a} = (3\cos\theta + 6)\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{b} = (\cos\theta - 2)\boldsymbol{i} + (1 + \sin\theta)\boldsymbol{j}$.

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular,

a. show that $3\sin^2\theta - 7\sin\theta + 2 = 0$; [3]

b. find the smallest possible positive value of θ. [3]

Given that $\boldsymbol{a} = 2\sin\theta\boldsymbol{i} + (1 - \sin\theta)\boldsymbol{j}$, find the value of the acute angle θ , so that $\boldsymbol{a}$ is perpendicular to the line $x + y = 1$.

A line L_1 has equation $\boldsymbol{r} = \begin{pmatrix} -5 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

A line L_2 passing through the origin intersects L_1 and is perpendicular to L_1.

(a) Find a vector equation of L_2.

(b) Determine the shortest distance from the origin to L_1.

Port A is defined to be the origin of a set of coordinate axes and port B is located at the point (70, 30), where distances are measured in kilometres.

A ship S_1 sails from port A at 10:00 in a straight line such that its position t hours after 10:00 is given by $r = t\begin{pmatrix} 10 \\ 20 \end{pmatrix}$.

A speedboat S_2 is capable of three times the speed of S_1 and is to meet S_1 by travelling the shortest possible distance. What is the latest time that S_2 can leave port B?

The coordinates of points A, B and C are given as $(5,\ -2,\ 5)$, $(5,\ 4,\ -1)$ and $(-1,\ -2,\ -1)$ respectively.

a. Show that AB = AC and that $\text{B}\hat{\text{A}}\text{C} = 60°$. [4]

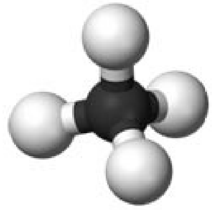
b. Find the Cartesian equation of Π, the plane passing through A, B, and C. [4]

c(i)(ii) (i) Find the Cartesian equation of Π_1 , the plane perpendicular to (AB) passing through the midpoint of [AB] . [4]

(ii) Find the Cartesian equation of Π_2 , the plane perpendicular to (AC) passing through the midpoint of [AC].

d. Find the vector equation of L , the line of intersection of Π_1 and Π_2 , and show that it is perpendicular to Π . [3]

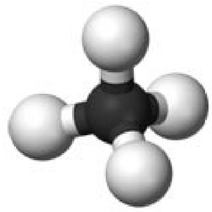
e. A methane molecule consists of a carbon atom with four hydrogen atoms symmetrically placed around it in three dimensions. [3]



The positions of the centres of three of the hydrogen atoms are A, B and C as given. The position of the centre of the fourth hydrogen atom is D.

Using the fact that $\text{AB} = \text{AD}$, show that the coordinates of one of the possible positions of the fourth hydrogen atom is $(-1,\ 4,\ 5)$.

f. A methane molecule consists of a carbon atom with four hydrogen atoms symmetrically placed around it in three dimensions. [6]



The positions of the centres of three of the hydrogen atoms are A, B and C as given. The position of the centre of the fourth hydrogen atom is D.

Letting D be $(-1,\ 4,\ 5)$, show that the coordinates of G, the position of the centre of the carbon atom, are $(2,\ 1,\ 2)$. Hence calculate $\text{D}\hat{\text{G}}\text{A}$, the bonding angle of carbon.

The equations of the lines L_1 and L_2 are

$$L_1 : r_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$L_2 : r_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.$$

a. Show that the lines L_1 and L_2 are skew. [4]

b. Find the acute angle between the lines L_1 and L_2. [4]

c. (i) Find a vector perpendicular to both lines. [10]

(ii) Hence determine an equation of the line L_3 that is perpendicular to both L_1 and L_2 and intersects both lines.

Find the acute angle between the planes with equations $x + y + z = 3$ and $2x - z = 2$.

The planes $2x + 3y - z = 5$ and $x - y + 2z = k$ intersect in the line $5x + 1 = 9 - 5y = -5z$.
Find the value of k .

(a) Write the vector equations of the following lines in parametric form.

$$r_1 = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + m \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + n \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

(b) Hence show that these two lines intersect and find the point of intersection, A.

(c) Find the Cartesian equation of the plane $\prod$ that contains these two lines.

(d) Let B be the point of intersection of the plane $\prod$ and the line $r = \begin{pmatrix} -8 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix}$.

Find the coordinates of B.

(e) If C is the mid-point of AB, find the vector equation of the line perpendicular to the plane $\prod$ and passing through C.

A plane π has vector equation $\boldsymbol{r} = (-2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) + \lambda(2\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}) + \mu(6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k})$.

(a) Show that the Cartesian equation of the plane π is $3x + 2y - 6z = 12$.

(b) The plane π meets the x, y and z axes at A, B and C respectively. Find the coordinates of A, B and C.

(c) Find the volume of the pyramid OABC.

(d) Find the angle between the plane π and the x-axis.

(e) **Hence**, or otherwise, find the distance from the origin to the plane π.

(f) Using your answers from (c) and (e), find the area of the triangle ABC.

The function f is defined on the domain [0, 2] by $f(x) = \ln(x + 1)\sin(\pi x)$.

a. Obtain an expression for $f'(x)$. [3]

b. Sketch the graphs of f and f' on the same axes, showing clearly all x-intercepts. [4]

c. Find the x-coordinates of the two points of inflexion on the graph of f. [2]

d. Find the equation of the normal to the graph of f where $x = 0.75$, giving your answer in the form $y = mx + c$. [3]

e. Consider the points A $(a\ ,\ f(a))$, B $(b\ ,\ f(b))$ and C $(c\ ,\ f(c))$ where a , b and c $(a < b < c)$ are the solutions of the equation $f(x) = f'(x)$. Find the area of the triangle ABC. [6]

The position vector at time t of a point P is given by

$$\overrightarrow{\mathrm{OP}} = (1+t)\,\boldsymbol{i} + (2-2t)\,\boldsymbol{j} + (3t-1)\,\boldsymbol{k},\ t \geqslant 0.$$

(a) Find the coordinates of P when $t = 0$.

(b) Show that P moves along the line L with Cartesian equations

$$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3}$$

(c) (i) Find the value of t when P lies on the plane with equation $2x + y + z = 6$.

(ii) State the coordinates of P at this time.

(iii) Hence find the total distance travelled by P before it meets the plane.

The position vector at time t of another point, Q, is given by

$$\overrightarrow{\mathrm{OQ}} = \begin{pmatrix} t^2 \\ 1-t \\ 1-t^2 \end{pmatrix},\ t \geqslant 0.$$

(d) (i) Find the value of t for which the distance from Q to the origin is minimum.

(ii) Find the coordinates of Q at this time.

(e) Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be the position vectors of Q at times $t = 0$, $t = 1$ and $t = 2$ respectively.

(i) Show that the equation $\boldsymbol{a} - \boldsymbol{b} = k\,(\boldsymbol{b} - \boldsymbol{c})$ has no solution for k .

(ii) Hence show that the path of Q is not a straight line.